PHYSICAL REVIEW E **83**, 057102 (2011)

Diversity and critical behavior in prisoner's dilemma game

C. K. Yun,[1] N. Masuda,[2,3] and B. Kahng[1,4]

[1]*Department of Physics and Astronomy, Seoul National University, Seoul 151-747, Korea*
[2]*Department of Mathematical Informatics, The University of Tokyo, Tokyo 113-8656, Japan*
[3]*PRESTO, Japan Science and Technology Agency, 4-1-8 Honcho, Kawaguchi, Saitama 332-0012, Japan*
[4]*School of Physics, Korea Institute for Advanced Study, Seoul 130-722, Korea*

(Received 15 June 2010; revised manuscript received 20 March 2011; published 25 May 2011)

The prisoner's dilemma (PD) game is a simple model for understanding cooperative patterns in complex systems. Here, we study a PD game problem in scale-free networks containing hierarchically organized modules and controllable shortcuts connecting separated hubs. We find that cooperator clusters exhibit a percolation transition in the parameter space (p,b), where p is the occupation probability of shortcuts and b is the temptation payoff in the PD game. The cluster size distribution follows a power law at the transition point. Such a critical behavior, resulting from the combined effect of stochastic processes in the PD game and the heterogeneity of complex network structure, illustrates diversities arising in social relationships and in forming cooperator groups in real-world systems.

DOI: 10.1103/PhysRevE.83.057102 PACS number(s): 89.75.Hc, 89.75.Da, 89.75.Fb

Social interactions between individuals are often cooperative or competitive, through which certain patterns such as the separation between cooperator groups and defector groups may emerge. In particular, social dilemmas in which the benefit of the whole society contradicts that of each individual has been an attractive topic of interdisciplinary research. The prisoner's dilemma (PD) game has been used as a basic model for social dilemmas [1,2]. In the two-player PD game, a player earns a larger payoff by unilateral defection than by mutual cooperation. Therefore, even if both players attain nothing when they defect, the optimal choice would be defection. In real systems, however, people are often cooperative and altruistic.

A social system may be described by a network of players [3]. If players in a network change their strategies according to evolutionary dynamics (i.e., by imitating successful neighbors), we often observe mutual cooperation and the formation of cooperator clusters. Aggregation of such clusters induces a phase transition [4,5]. Such an emerging pattern results from the collective dynamics of players' interactions with each other [6]. While such a pattern is present and probably functional in social relationships, the diversity in the cluster size distribution has not been clearly understood yet. In this Brief Report, we investigate the formation of diverse sizes of cooperator groups in the PD game as the network topology changes from large-world to small-world network.

In the context of networks, the PD game were firstly studied in the Euclidean space [1,7]. Then, it has been studied in complex networks such as scale-free (SF) networks, in which the degree distribution follows a power law. In random SF networks such as the Barabási and Albert (BA) model [8], the mean distance between two nodes scales logarithmically with system size N. In such small-world SF networks, the density of cooperators is significantly enhanced [9] compared with that in the Euclidean space and cooperators can be stable. The stabilization of cooperation is initiated from the hubs and then spreads to nodes with smaller degrees [9,10]. Thus, the hub plays a crucial role in spreading cooperation.

Many SF networks in the real world are not as random as that in the BA model; they contain modular structure within them. Moreover, the modular structure is hierarchically organized [11]. In such modular SF networks, hubs are separated from each other, and the mean distance between two nodes often scales in a power-law manner with the system size [12]. Such networks are called large-world or fractal networks. Cooperation in such networks is less than that in random SF networks, because the large distance between hubs generally reduces cooperation [9,10].

Social networks in the real world are often at the boundary between small-world and large-world networks [13]. Thus, in this Brief Report, we study the PD game in artificial networks in which the number of edges between separated hubs is controlled by the occupation probability p and examine the diversity occurring in the cooperator cluster sizes as the network transforms from large-world to small-world network as p increases. We find that clusters composed of cooperators undergo a percolation transition in the parameter space of (p,b), where b is the temptation payoff in the PD game. Interestingly, the percolation transition occurs either continuously or discontinuously as p is increased for a fixed b or b is decreased for a fixed p. Therefore, there exists a tricritical-like point (p_t,b_t) such that for a fixed $b < b_t$ $(b > b_t)$ or $p < p_t$ $(p > p_t)$, the giant cluster of cooperators grows continuously (discontinuously) [Fig. 1(a)]. The phase diagram is shown in Fig. 1(b). Furthermore, the size distribution of the cooperator clusters exhibits a power-law behavior near the percolation threshold as long as $p < p_t$. This result suggests that the cooperators organize their clusters to attain a critical state, which enhances the diversity within the system.

On the basis of previous papers [5,9,10], we define the payoff matrix by

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix} \end{array}, \tag{1}$$

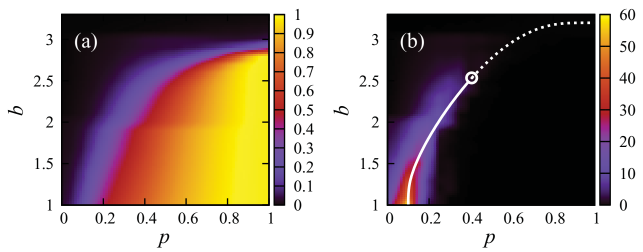

FIG. 1. (Color online) (a) Giant cluster size in the parameter space of (p,b) for the hierarchical network with system size $N = 10{,}924$. Plotted data are averages over 100 network configurations. (b) Susceptibility under the same condition as that in the case of (a). The solid and dotted curves representing continuous and discontinuous percolation transitions, respectively, are the loci along the peaks of susceptibility.

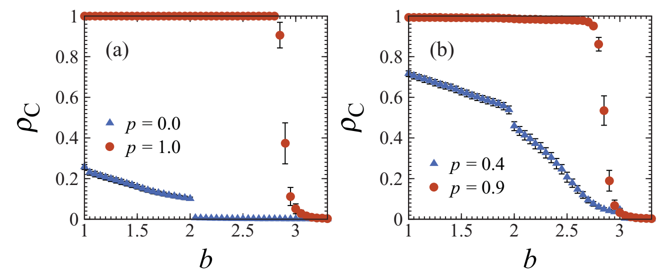

FIG. 2. (Color online) Density of cooperators as a function of temptation payoff b for the cases (a) $p = 0$ and $p = 1$ and (b) $p = 0.4$ and 0.9 in the hierarchical model.

corresponding to the weak PD game payoff matrix. A row player selects one of two states, either cooperation (C) or defection (D), and so does a column player. Each entry in the matrix (1) represents the payoff that the row player obtains. Regardless of whether the opponent selects C or D, both players are tempted to defect in order to obtain a larger payoff (temptation) i.e., $b > 1$. If both players act selfishly to defect, both obtain no payoff. However, the individual rewards for both players could be higher if they mutually cooperate.

We examine the evolutionary PD game for several types of complex networks with equal initial densities of C and D. At each time step (or round), each player i interacts with all of its k_i neighbors; here, k_i is the degree of the node i. Player i's payoff in one round P_i is the sum of all the payoffs earned by playing against the k_i neighbors. Player i updates its strategy according to the following rule [9]: A neighbor j of player i is chosen with equal probability $1/k_i$. Then, if $P_j > P_i$, i copies j's strategy with probability $(P_j - P_i)/[b\max(k_i,k_j)]$. The denominator normalizes the probability such that the probability is between 0 and 1. On the other hand, if $P_j < P_i$, i does not change its strategy. The rule for updating strategies is synchronously applied by all the players. This procedure is repeated in subsequent rounds. We find that the main result, a power-law behavior in the size distribution of cooperator clusters, obtained from the parallel updating rule does not change even when updating is performed sequentially [9].

In each round, a player chooses either C or D. However, in long time limit, players are categorized as [10] a permanent cooperator, a permanent defector, or an unstable player that continually changes its state between C and D. The situation created by unstable players, i.e., the alternative changing between cooperation and defection, may be called cooperative solution. We simulate the PD game until the density of the cooperators does not change with time. We simulate the PD game for up to 2×10^4 rounds after a steady state is reached for each initial configuration. A permanent cooperator (defector) is defined as the node that chooses only C (D) for the last 10^4 rounds. The rest of the nodes are regarded as unstable players. The density of the permanent cooperators is denoted by ρ_c.

The PD game is played on the so-called hierarchical network introduced in [14]. This network is constructed by repeating a simple structural mapping as follows: We begin with an edge, which is replaced with four edges formed in a diamond shape in the next iteration step. A horizontal edge between two nodes at the diagonal positions is present with probability p. This process is repeated for each edge in the diamond (except the horizontal bond) in subsequent iterations until the obtained network attains the desired system size. This network is scale-free and has a degree exponent of 3. When $p = 0$, the network is a large-world network and the diameter, i.e., the largest distance between any two nodes in the system, increases according to a power law in terms of the system size N. When $p = 1$, the network is a small-world network and the diameter is proportional to $\ln N$.

The density of the permanent cooperators ρ_c is shown as a function of b for various p values in Fig. 2. When $p = 1$, the cooperator density ρ_c is almost equal to 1 for b values up to b_c, beyond which ρ_c drops suddenly. When $p = 0$, ρ_c is relatively small even when $b = 1$, and it decays continuously with increasing b. When $p = 0$, a player i with degree $\geqslant 4$ is surrounded by players with degree 2 (see Fig. 3 for the

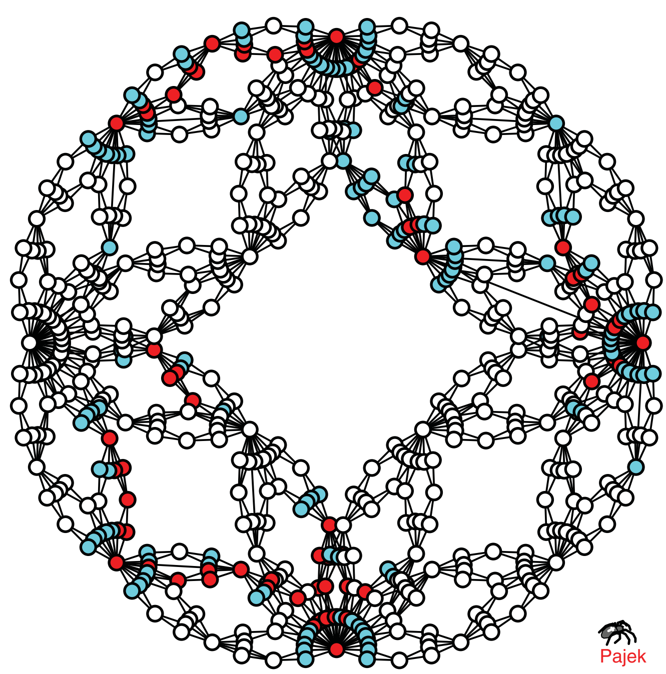

FIG. 3. (Color online) A snapshot of permanent cooperators (red, dark), permanent defectors (white), and unstable players (cyan, gray) in the hierarchical network with $p = 0.15$, $b = 2.3$, and $N = 684$ after 20,000 rounds in a steady state.

network structure). A degree-2 defector j adjacent to i obtains payoff $\geqslant b$ if i cooperates. The payoff of cooperator i is equal to the number of degree-2 cooperators adjacent to i, which is an integer. Therefore, whether the cooperator is likely to invade or is invaded along the link between i and j would drastically change when b crosses integer values. This is the case for any link because of the deterministic structure of the network. Therefore, we observed a discontinuous jump at $b = 2$ in Fig. 2. However, such behavior does not appear in nondeterministic networks such as real-world networks. It is noteworthy that for all b, ρ_c for $p = 1$ is larger than ρ_c for $p = 0$. This indicates that the shortcuts connecting the hubs play an important role in enhancing cooperation [9,10]. The cooperation between influential individuals enhances the overall cooperation in the society.

A snapshot of the states of the players when $p = 0.15$, $b = 2.3$, and $N = 684$ is shown in Fig. 3: this network represents a large-world network. The permanent cooperators tend to locate around or at hubs, and may impact other cooperators. This is because once cooperators form a cluster around a hub, then the hub becomes stable and is protected against invasion by defectors. However, when defectors gather around a hub cooperator, the hub may cease to be permanently stable because defectors get higher payoff than the hub cooperator. Nevertheless, permanent defectors tend to be located at nodes with small degree, and the unstable players in between nodes with small degrees and hubs.

On the other hand, we remark that cooperators are not *always* located at or around hubs as they were in the snapshot, because the formation of cooperator clusters is determined stochastically and it generally depends on the fluctuations in cooperator densities. The heterogeneity of the degree of the nodes and the stochastic process of the PD game result in the formation of cooperator clusters with a wide range of sizes.

In the small-world network with a large p, however, permanent cooperators always locate at hubs and mutual cooperation occurs on a global scale. This behavior occurs even when a hub and its neighbors are mostly defectors at an early stage; the hub eventually cooperates with another hub.

The size $G(p,b)$ of the largest cluster of permanent cooperators per the system size is shown as a function of p and b in Fig. 1(a). Here $G(p,b)$ is considered to be the order parameter, as in percolation theory. As p increases for a fixed $b < b_t$ (b_t is defined below), G increases gradually from 0 to 1 and exhibits a percolation transition at $p_c(b)$. In Fig. 1(a), we can see that the transition interval in which $0.1 < G < 0.4$ is wide for small b. However, this interval becomes narrower as b increases, indicating that the giant cluster grows suddenly as p increases for large values of b. A similar behavior appears in the susceptibility, defined by $\chi(p,b) = \sum_s' s^2 n_s$, where the prime denotes the exclusion of the giant cluster in the summation, and n_s is the number of s-sized clusters relative to the system size composed of permanent cooperators. As shown in Fig. 1(b), the peak positions of χ mark the phase boundary $p_c(b)$ across which the giant cooperator cluster increases to a macroscopic-scale cluster. The peak height of the susceptibility decreases as b approaches the tricritical-like point b_t, beyond which the susceptibility is likely to disappear. It is noteworthy that while the peak diverges in the classical percolation transition, it is

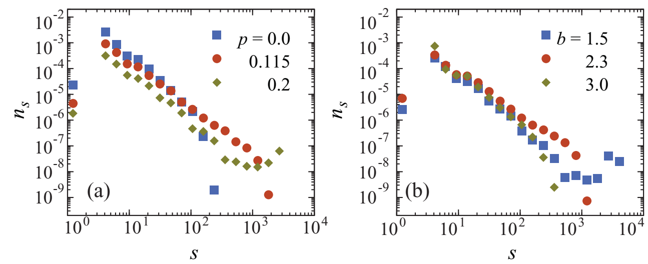

FIG. 4. (Color online) Size distribution of cooperator clusters in the hierarchical network with $N = 10{,}924$ nodes for (a) various values of p and $b = 1.7$ and (b) various values of b and $p = 0.15$. Data are averages over 500 network configurations.

finite in the infinite-order percolation transition, which often occurs in growing networks [15]. On the basis of this fact, we may say that the percolation transition is continuous for $p < p_t$ and of infinite order for $p > p_t$. The estimated value of p_t is about 0.4; this is roughly equal to $p^* \simeq 0.494$, which is the boundary between the large-world and the small-world networks as determined on the basis of the thermal transition patterns of the Ising model [14].

In Fig. 4(a), the cluster-size distribution n_s is plotted against s for several p and $b = 1.7$. In Fig. 4(b), n_s is plotted for several b and $p = 0.15$. For the large-world network, i.e., the network for which $p = 0.0 < p_c$, $n_s(p)$ exhibits a subcritical behavior, i.e., for small s, it decays according to a power law and for large s, it decays exponentially beyond a cutoff. At $p_c \cong 0.1$, $n_s(p_c)$ obeys the power law $n_s(p_c) \sim s^{-\tau}$ with $\tau \approx 1.85 \pm 0.1$. When $p > p_c$, $n_s(p)$ exhibits supercritical behavior. For the small-world network, the cluster-size distribution does not follow a power law. Similar behavior is observed when p is kept constant while varying b [Fig. 4(b)]. When $p < p_t$, the order parameter increases gradually with b, while when $p > p_t$, it increases very drastically.

Large-world networks are known as fractal networks. Typical fractal networks in the real world are the protein interaction network and the World-Wide Web (WWW) [12]. In this study, we simulate the evolutionary PD game on the undirected version of the WWW. Since the WWW network is a single network, corresponding to p being fixed, we vary only b. As shown in Fig. 5, the distribution of cooperator cluster sizes for several values of b decays according to a power law with an exponent of approximately 2. Thus, we conclude

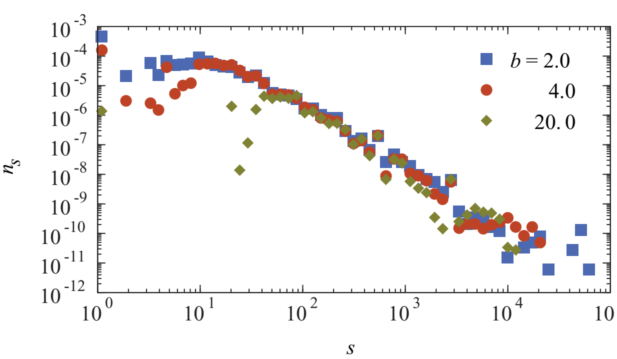

FIG. 5. (Color online) Size distribution of cooperator clusters for several values of b in the WWW with system size $N = 325{,}729$.

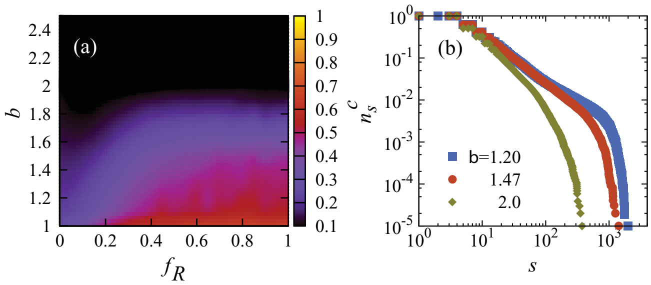

FIG. 6. (Color online) (a) Giant cluster size in the parameter space of (f_R,b) for rewired networks with $N = 10{,}924$, where f_R is the fraction of rewired links. Data are averages over 100 network configurations. (b) Accumulated size distribution of cooperator clusters at $f_R = 0.01$ for several values of b.

that the critical behavior of the cluster-size distribution is not limited to the hierarchical networks, but rather, this behavior is intrinsic.

Since PD game dynamics generally depends on the number of links of a network, one may wonder if our results for different p values [Fig. 1 and Fig. 4(a)] are intrinsic. Thus, we construct rewired networks in which the degree of each node remains the same but f_R fraction of links are rewired. While the giant cluster size $G(f_R,b)$ looks somewhat different in Fig. 6(a), the distribution of cooperator cluster sizes exhibits a critical behavior at a certain value b_c for a given f_R [Fig. 6(b)].

In summary, we have studied the percolation transition of cooperator clusters in fractal hierarchical networks. We found that in the WWW, the cluster-size distribution of permanent cooperators follows a power law near the percolation threshold. Such a critical behavior is also observed in the artificial hierarchical networks. The power-law behavior indicates that cooperators create communities of diverse sizes at scattered locations. These clusters stochastically form. In order to improve cooperations on a global scale in the society, communication channels must be established between influential individuals.

This study was supported by an NRF grant (Grant No. 2010-0015066) and the NAP of KRCF (BK), and by Grants-in-Aid for Scientific Research (Nos. 20760258 and 20540382) from MEXT, Japan (NM).

[1] R. Axelrod, *Evolution of Cooperation* (New York, NY: Basic Books, 1984).

[2] M. A. Nowak, *Evolutionary Dynamics—Exploring the Equations of Life* (The Belknap Press of Harvard University Press, Cambridge, MA, USA 2006).

[3] M. E. J. Newman, SIAM Rev. **45**, 167 (2003); D. J. Watts, Annu. Rev. Sociol. **30**, 243 (2004); S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Phys. Rep. **424**, 175 (2006); C. Castellano, S. Fortunato, and V. Loreto, Rev. Mod. Phys. **81**, 591 (2009).

[4] G. Szabó and C. Tőke, Phys. Rev. E **58**, 69 (1998).

[5] G. Szabó and G. Fáth, Phys. Rep. **446**, 97 (2007).

[6] F. C. Santos, M. D. Santos, and J. M. Pacheco, Nature (London) **454**, 213 (2008).

[7] M. A. Nowak and R. M. May, Nature (London) **359**, 826 (1992).

[8] A. L. Barabási and R. Albert, Science **286**, 509 (1999).

[9] F. C. Santos and J. M. Pacheco, Phys. Rev. Lett. **95**, 098104 (2005); F. C. Santos, J. F. Rodrigues, and J. M. Pacheco, Proc. R. Soc. B **273**, 51 (2006).

[10] J. Gómez-Gardeñes, M. Campillo, L. M. Floría, and Y. Moreno, Phys. Rev. Lett. **98**, 108103 (2007).

[11] E. Ravasz, A. L. Somera, D. A. Mongru, Z. N. Oltvai, and A.-L. Barabási, Science **297**, 1551 (2002).

[12] C. Song, S. Havlin, and H. A. Makse, Nature (London) **433**, 392 (2005); K. I. Goh, G. Salvi, B. Kahng, and D. Kim, Phys. Rev. Lett. **96**, 018701 (2006).

[13] D. Lee, K. I. Goh, B. Kahng, and D. Kim, Phys. Rev. E **82**, 026112 (2010).

[14] M. Hinczewski and A. N. Berker, Phys. Rev. E **73**, 066126 (2006).

[15] D. S. Callaway, J. E. Hopcroft, J. M. Kleinberg, M. E. J. Newman, and S. H. Strogatz, Phys. Rev. E **64**, 041902 (2001).